

A BORDISM APPROACH TO STRING TOPOLOGY

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ABSTRACT. Using intersection theory in the context of Hilbert manifolds and geometric homology we show how to recover the main operations of string topology built by M. Chas and D. Sullivan. We also study and build an action of the homology of reduced Sullivan's chord diagrams on the singular homology of free loop spaces, extending previous results of R. Cohen and V. Godin and unifying part of the rich algebraic structure of string topology as an algebra over the prop of these reduced diagrams. Some of these operations are extended to spaces of maps from a sphere to a compact manifold.

1. INTRODUCTION

The study of spaces of maps is an important and difficult task of algebraic topology. In this paper we study n -sphere spaces, they are topological spaces of unbased maps from a n -sphere into a manifold. Our aim is to study the algebraic structure of the homology of these spaces. Let us begin by giving some motivations for the study of such spaces. We focus on free loop spaces ($n = 1$).

The study of free loop spaces over a compact oriented manifold plays a central role in algebraic topology. There is a non exhaustive list of topics where free loop spaces appear. Let us review some of them:

- Free loop spaces are one of the main tool in order to study closed geodesics on Riemannian manifolds. Let M be Riemannian, compact, connected, simply connected, of dimension greater than one. D. Gromoll and W. Meyer proved that there exist infinitely many (geometrically distinct) periodic geodesics on an arbitrary Riemannian manifold if the Betti numbers of the free loop space of M are unbounded [25]. And using methods of rational homotopy theory M. Vigué and D. Sullivan showed that these rational Betti numbers are unbounded if and only if the rational cohomology has at least two generators [51].
- According to works of D. Burghelea and Z. Fiedorowicz [7] of T. Goodwillie [24] and of J.D.S. Jones [31], the cohomology of free loop spaces is strongly related to Hochschild homology and the S^1 -equivariant cohomology of free

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loop spaces is also related to cyclic homology and over a field of characteristic zero to Waldhausen algebraic K -theory. Hence rationally, the homotopy of free loop spaces is related to the space of automorphisms of manifolds (via Waldhausen's theory). To go further in this direction there are also some relations between the suspension spectrum of the free loop spaces and topological cyclic homology [5].

- Analysis on such spaces has always been a source of constant inspiration, for example one can cite the work of E. Witten on Index of Dirac operators on Free loop spaces [54], this was the beginning of the theory of elliptic genera and elliptic cohomology [46]. This shed new lights on the periodicity phenomena in stable homotopy and the analysis underlying it [1].

- One can also cite the work of K. T. Chen who has introduced a chain complex based on iterated integrals that computes the cohomology of free loop spaces over a manifold [10]. Iterated integrals give a De-Rham theory for path spaces. In particular, this theory has found some applications in the algebraic interpretation of index theory on free loop spaces by means of cyclic homology [20] and [22]. Let us notice that the theory of iterated integrals is also related to algebraic geometry (see [26] for a survey).

More recently the discovery by M. Chas and D. Sullivan of a Batalin Vilkovisky structure on the singular homology of these spaces [8] had a deep impact on the subject and has revealed a part of a very rich algebraic structure [9], [12]. The BV -structure consists in

- A loop product $-\bullet-$ which is commutative and associative, it can be understood as an attempt to perform intersection theory of families of closed curves,
- A loop bracket $\{-, -\}$, half of this bracket controls the commutativity up to homotopy at the chain level of the loop product,
- An operator Δ coming from the action of S^1 on the free loop space (S^1 acts by reparametrization of the loops).

M. Chas and D. Sullivan use (in [8]) "classical intersection theory of chains in a manifold". This structure has also been defined in a purely homotopical way by R. Cohen and J. Jones using a ring spectrum structure on a Thom spectrum of a virtual bundle over free loop spaces [13]. As discovered by S. Voronov [53], it comes in fact from a geometric operadic action of the cacti operad. Very recently J. Klein in [33] has extended the homotopy theoretic approach of R. Cohen and J. Jones to Poincaré duality spaces using A_∞ -ring spectrum technology.

In this paper we adopt a different approach to string topology, namely we use a geometric version of singular homology introduced by M. Jakob [28]. And we show how it is possible to define Gysin morphisms, exterior products and intersection type products (such as the loop product of M. Chas and D. Sullivan) in the setting of Hilbert manifolds. Let us point out that three different types of free loop spaces are used in the mathematical literature:

- Spaces of continuous loops ([8] for example),
- Spaces of smooth loops, which are Fréchet manifolds but not Hilbert manifolds ([6] for some details),
- Spaces of Sobolev class of loops [34] or [25],

These three spaces are very different from an analytical point of view, but they are homotopy equivalent. For our purpose, we deal with Hilbert manifolds in order to have a nice theory of transversality, the last of maps is the one we use.

In order to perform such intersection theory we recall in section 2 what is known about transversality in the context of Hilbert manifolds. We also describe the manifold structure of free loop spaces used by W. Klingenberg [34] in order to study closed geodesics on Riemannian manifolds. The cornerstone of all the constructions of the next sections will be the "string pull-back", also used by R. Cohen and J. Jones [13, diagram 1.1]. In section 2.4 we extend these techniques n -sphere spaces and we show how to intersect geometrically families of n -sphere in M .

Section 3 is devoted to the introduction and main properties of geometric homology. This theory is based upon bordism classes of singular manifolds. In this setting families of n -sphere in M , which are families parametrized by smooth oriented compact manifolds, have a clear homological meaning. Of particular interest and crucial importance for applications to sphere topology is the construction of an explicit Gysin morphism for Hilbert manifolds in the context of geometric homology (section 3.3). This construction does not use any Thom spaces and is based on the construction of pull-backs for Hilbert manifolds. Such approach seems completely new in this context.

We want to point out that all the constructions performed in this section work with a generalized homology theory h_* under some mild assumptions ([28]):

- the associated cohomology theory h^* is multiplicative,
- h_* satisfies the infinite wedge axioms.

In section 4 the operator Δ , the loop product, the loop bracket, the intersection morphism and the string bracket are defined and studied using the techniques introduced in section 2 and 3. This section is also concerned with string topology operations, these operations are parametrized by the topological space of reduced Sullivan's Chord diagrams $\overline{\mathcal{CF}_{p,q}^\mu(g)}$, which is closely related to the combinatorics of Riemann surfaces of genus g , with p -incoming boundary components and q -outgoing. A. Voronov (private communication) suggested to introduce these spaces of diagrams because they form a prop and the cacti appear as a sub-operad. Pushing the work of R. Cohen and V. Godin on the cation of Sullivan's chord diagrams on free loop spaces further we prove:

Theorem: *Let \mathcal{LM} be the free loop space over a compact d dimensional manifold M . For $q > 0$ there exist morphisms:*

$$\mu_{n,p,q}(g) : H_n(\overline{\mathcal{CF}_{p,q}^\mu(g)}) \rightarrow \text{Hom}(H_*(\mathcal{LM}^{\times p}), H_{*+\chi(\Sigma).d+n}(\mathcal{LM}^{\times q})).$$

where $\chi(\Sigma) = 2 - 2g - p - q$.

Moreover as these operations are compatible with the gluing of reduced Sullivan's diagrams, $H_*(\mathcal{LM})$ appear as an algebra over this Prop. As a corollary one recovers the structure of a Frobenius algebra on $H_{*+d}(\mathcal{LM})$, build in [12], the operator Δ and the loop product of M. Chas and D. Sullivan [8].

Section 5 is to devoted to the extension of the results of the section 4 to n -sphere spaces. In particular, there exists a commutative and associative product on the homology of these spaces. The case of 3-sphere spaces is detailed.

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2. INFINITE DIMENSIONAL MANIFOLDS

2.1. Recollections on Hilbert manifolds. This section is expository, we review the basic facts about Hilbert manifolds, we refer to [39] (see also [37] for a general introduction to infinite dimensional manifolds). Moreover all the manifolds we consider in this paper are Hausdorff and second countable (we need these conditions in order to consider partitions of unity).

2.1.1. Differential calculus. Let E and F be two topological vector spaces, there is no difficulty to extend the notion of differentiability of a continuous map between E and F . Hence, let $f : E \rightarrow F$ be a continuous map we say that f is differentiable at $x \in E$ if for any $v \in E$ the limit:

$$df_x(v) = \lim_{t \rightarrow 0} \frac{f(x + tv) - f(x)}{t}$$

exists. One can define differentials, C^∞ morphisms, diffeomorphisms and so on.

2.1.2. *Hilbert manifolds.* A topological space X is a manifold modelled on a separable Hilbert space E if there exists an atlas $\{U_i, \phi_i\}_{i \in I}$ such that:

- i) each U_i is an open set of X and $X = \bigcup_{i \in I} U_i$,
- ii) $\phi_i : U_i \rightarrow E$ is an homeomorphism,
- iii) $\phi_i \phi_j^{-1}$ is a diffeomorphism whenever $U_i \cap U_j$ is not empty.

2.1.3. *Fredholm maps.* A smooth map $f : X \rightarrow Y$ between two Hilbert manifolds is a Fredholm map if for each $x \in X$, the linear map

$$df_x : T_x X \longrightarrow T_{f(x)} Y$$

is a Fredholm operator, that is to say if $\ker df_x$ and $\text{coker } df_x$ are finite dimensional vector spaces. Recall that the index of a Fredholm map

$$\text{index} : X \rightarrow \mathbb{Z}$$

$$\text{index}(f_x) = \dim(\ker df_x) - \dim(\text{coker } df_x)$$

is a continuous map.

2.1.4. *Orientable morphisms.* A smooth map $f : X \rightarrow Y$ between two Hilbert manifolds is an orientable morphism if it is a proper map (the pre-image of a compact set is compact) which is also Fredholm and such that the normal bundle $\nu(f)$ is orientable (for convenience we consider the notion of orientability with respect to singular homology but we could have chosen to work in a much more general setting).

Let us remark that a closed embedding is an orientable morphism if and only if $\nu(f)$ is finite dimensional and orientable. A closed Fredholm map is proper by a result of S. Smale [47].

2.1.5. *Partitions of unity.* A very nice feature of Hilbert manifolds with respect to Banach manifolds and other type of infinite dimensional manifolds is the existence of partitions of unity ([39, chapter II,3] for a proof). As a consequence mimicking techniques used in the finite dimensional case, one can prove that any continuous map

$$f : P \rightarrow X$$

from a finite dimensional manifold P to an Hilbert manifold X is homotopic to a C^∞ one. And we can also smoothen homotopies.

2.2. **Transversality.** We follow the techniques developed by A. Baker and C. Öznel in [2] in order to deal with transversality in an infinite dimensional context.

2.2.1. *Transversal maps.* Let $f : X \rightarrow Y$ and $g : Z \rightarrow Y$ be smooth maps between two Hilbert manifolds. Then they are transverse at $y \in Y$ if

$$df(T_x X) + dg(T_z Z) = T_y Y$$

with $f(x) = g(z) = y$. The maps are transverse if they are transverse at any point $y \in \text{Im } f \cap \text{Im } g$.

2.2.2. *Pull-backs.* Let us recall the main results about pull-backs of Hilbert manifolds. We consider the following diagram:

$$\begin{array}{ccc}
 & Z \xleftarrow{g^*f} Z \cap_Y X & \\
 g \downarrow & & \downarrow \phi \\
 Y \xleftarrow{f} X & &
 \end{array}$$

where Z is a finite dimensional manifold and $f : X \rightarrow Y$ is an orientable map.

Using an infinite dimensional version of the implicit function theorem [39, Chapter I,5], one can prove the following result:

2.2.3. **Proposition.** [2, prop. 1.17] *If the map*

$$f : X \rightarrow Y$$

is an orientable morphism and

$$g : Z \rightarrow Y$$

is a smooth map transverse to f , then the pull-back map:

$$g^*f : Z \cap_Y X \longrightarrow Z$$

is an orientable morphism.

Moreover the finite dimensional hypothesis on Z enables to prove:

2.2.4. **Theorem.** [2, Th. 2.1, 2.4] *Let*

$$f : X \rightarrow Y$$

be an orientable morphism and let

$$g : Z \rightarrow Y$$

be a smooth map from a finite dimensional manifold Z . Then g can be deformed by a smooth homotopy until it is transverse to f .

2.3. **Free loop spaces.** If we want to do intersection theory with spaces of closed curves, we need to consider them as smooth manifolds. Following [6, Chapter 3], one can consider the space $C^\infty(S^1, M)$ of all smooth curves as an Inverse Limit Hilbert manifold. But we prefer to enlarge this space and to consider $\mathbf{H}^1(S^1, M)$ the space of \mathbf{H}^1 curves. This space has the advantage to be an Hilbert manifold. With this choice we can apply all the techniques described in the sections 2.1 and 2.2.

In fact, we could also consider $\mathbf{H}^n(S^n, M)$ the space of \mathbf{H}^n -curves, all these spaces are Hilbert manifolds, and they are also all homotopy equivalent

to the ILH manifold $C^\infty(S^1, M)$. And all these manifolds are homotopy equivalent to the space of continuous maps $C^0(S^1, M)$ equipped with the compact open topology.

2.3.1. Manifold structure. In order to define an Hilbert manifold structure on free loop spaces we follow W. Klingenberg's approach [34].

Let M be a simply-connected Riemannian manifold of dimension d . We set

$$\mathcal{L}M = \mathbf{H}^1(S^1, M).$$

The manifold $\mathcal{L}M$ is formed by the continuous curves $\gamma : S^1 \rightarrow M$ of class \mathbf{H}^1 , it is modelled on the Hilbert space $\mathcal{L}\mathbb{R}^d = \mathbf{H}^1(S^1, \mathbb{R}^d)$. The space $\mathcal{L}\mathbb{R}^d$ can be viewed as the completion of the space $C_p^\infty(S^1, \mathbb{R}^d)$ of piecewise differentiable curves with respect to the norm $\| - \|_1$. This norm is defined via the scalar product:

$$\langle \gamma, \gamma' \rangle_1 = \int \gamma(t) \diamond \gamma'(t) dt + \int \delta\gamma(t) \diamond \delta\gamma'(t) dt,$$

where \diamond is the canonical scalar product of \mathbb{R}^d . As S^1 is 1-dimensional, we notice that by Sobolev's embedding theorem elements of $\mathcal{L}\mathbb{R}^d$ can be represented by continuous curves.

Let us describe an atlas $\{U_\gamma, \phi_\gamma\}$ of $\mathcal{L}M$. We take $\gamma \in C_p^\infty(S^1, M)$ a piecewise differentiable curve in M (notice that $C_p^\infty(S^1, M) \subset \mathbf{H}^1(S^1, M)$) and consider the pullback:

$$\begin{array}{ccc} \gamma^* TM & \longrightarrow & TM \\ \downarrow & & \downarrow \\ S^1 & \xrightarrow{\gamma} & M. \end{array}$$

Now let $T_\gamma^\epsilon \subset \gamma^* TM$ be the set of vectors of norm less than ϵ . The exponential map

$$\exp : T_\gamma^\epsilon \longrightarrow M$$

identifies T_γ^ϵ with an open set of M and induces a map:

$$\mathcal{L}T_\gamma^\epsilon \longrightarrow \mathcal{L}M.$$

moreover as $\gamma^* TM$ is a trivial vector bundle (because M is 1-connected), we fix a trivialization:

$$\varphi : \gamma^* TM \longrightarrow S^1 \times \mathbb{R}^d$$

this gives a chart:

$$\phi_\gamma : \mathcal{L}T_\gamma^\epsilon \longrightarrow \mathcal{L}\mathbb{R}^d.$$

2.3.2. Remark. In fact, the manifold structure on $\mathcal{L}M$ does not depend on a choice of a particular Riemannian metric on M .

2.3.3. *The tangent bundle.* Let $TM \rightarrow M$ be the tangent bundle of M . The tangent bundle of \mathcal{LM} denoted by $T\mathcal{LM}$ can be identified with $\mathcal{L}TM$, this is an infinite dimensional vector bundle where each fiber is isomorphic to the Hilbert space $\mathcal{L}\mathbb{R}^d$. Let $\gamma \in \mathcal{LM}$ we have:

$$T\mathcal{LM}_\gamma = \Gamma(\gamma^*TM),$$

where $\Gamma(\gamma^*TM)$ is the space of sections of the pullback of the tangent bundle of M along γ (this is the space of \mathbf{H}^1 vector fields along the curve γ). A trivialization φ of γ^*TM induces an isomorphism:

$$T\mathcal{LM}_\gamma \cong \mathcal{L}\mathbb{R}^d.$$

The tangent bundle of \mathcal{LM} has been studied in [14] and [43].

2.3.4. *Riemannian structure.* The manifold \mathcal{LM} has a natural Riemannian metric, the scalar product on $T\mathcal{LM}_\gamma \cong \mathcal{L}\mathbb{R}^d$ comes from $\langle -, - \rangle_1$.

2.3.5. *The S^1 -action.* The circle acts on \mathcal{LM} :

$$\Theta : S^1 \times \mathcal{LM} \longrightarrow \mathcal{LM}$$

by reparametrization:

$$\Theta(\theta, \gamma) : t \mapsto \gamma(t + \theta).$$

Of course this action is not free, it is continuous but not differentiable. Let $\gamma \in \mathcal{LM}$, then the isotropy subgroup $Is(\gamma)$ of γ is S^1 if and only if γ is a constant map, in this case we say that γ is of multiplicity 0. Otherwise it is isomorphic to a finite cyclic group and the multiplicity of the curve is equal to the order of the isotropy subgroup. Let $\mathcal{LM}^{(m)}$ be the space of curves of multiplicity equal to m . This gives an S^1 -equivariant partition of \mathcal{LM} :

$$\mathcal{LM} = \bigcup_m \mathcal{LM}^{(m)}.$$

The space $\mathcal{LM}^{(0)}$ can be identified with M , and $\mathcal{LM}^{(1)}$ is called the space of prime curves.

2.3.6. *The string pullback.* Let us consider the evaluation map

$$ev_0 : \mathcal{LM} \rightarrow M$$

$$\gamma \mapsto \gamma(0),$$

this is a submersion of Hilbert manifolds (this follows immediately from the definition of $T\mathcal{LM}$). As the map $ev_0 \times ev_0$ is transverse to the diagonal map Δ (because $ev_0 \times ev_0$ is a submersion), we can form the *string pull-back* [13, (1.1)]:

$$\begin{array}{ccc}
\mathcal{L}M \times \mathcal{L}M & \xleftarrow{\tilde{\Delta}} & \mathcal{L}M \cap_M \mathcal{L}M \\
\downarrow ev_0 \times ev_0 & & \downarrow ev \\
M \times M & \xleftarrow{\Delta} & M,
\end{array}$$

by transversality this is a diagram of Hilbert manifolds. We have:

$$\mathcal{L}M \cap_M \mathcal{L}M = \{(\alpha, \beta) \in \mathcal{L}M \times \mathcal{L}M / \alpha(0) = \beta(0)\}.$$

The map

$$\tilde{\Delta} : \mathcal{L}M \cap_M \mathcal{L}M \rightarrow \mathcal{L}M \times \mathcal{L}M$$

is a closed embedding of codimension d .

As the normal bundle $\nu_{\tilde{\Delta}}$ is the pull-back of ν_{Δ} and as this last one is isomorphic to TM , we deduce that $\tilde{\Delta}$ is an orientable morphism.

2.3.7. Families of closed strings. A family of closed strings in M is a smooth map

$$f : P \rightarrow \mathcal{L}M$$

from a compact orientable manifold P .

The proposition below gives a necessary but not sufficient condition in order to do intersection of families of closed strings.

2.3.8. Proposition. *If $P \times Q \xrightarrow{f \times g} \mathcal{L}M \times \mathcal{L}M$ is transverse to $\tilde{\Delta}$ then $ev_0 f$ and $ev_0 g$ are transverse in M .*

Now we suppose that (P, f) and (Q, g) are two orientable compact manifolds of dimensions p and q respectively. Moreover we suppose that they are such that $f \times g$ is transverse to $\tilde{\Delta}$. We denote by $P * Q$ the pullback:

$$\begin{array}{ccc}
P \times Q & \xleftarrow{\quad} & P * Q \\
\downarrow f \times g & & \downarrow \psi \\
\mathcal{L}M \times \mathcal{L}M & \xleftarrow{\tilde{\Delta}} & \mathcal{L}M \cap_M \mathcal{L}M.
\end{array}$$

Then $P * Q$ is a compact orientable submanifold of $P \times Q$ of dimension $p + q - d$.

2.3.9. *Composing loops.* Let us define the map:

$$\Upsilon : \mathcal{L}M \cap_M \mathcal{L}M \longrightarrow \mathcal{L}M.$$

Let (α, β) be an element of $\mathcal{L}M \cap_M \mathcal{L}M$ then $\Upsilon(\alpha, \beta)$ is the curve defined by:

$$\begin{aligned}\Upsilon(\alpha, \beta)(t) &= \alpha(2t) \text{ if } t \in [0, 1/2] \\ \Upsilon(\alpha, \beta)(t) &= \beta(2t - 1) \text{ if } t \in [1/2, 1].\end{aligned}$$

We notice that this map is well defined because we compose piecewise differential curves, hence no "dampening" constructions are needed as in [13, remark about construction (1.2)].

The construction of Υ comes from the co-H-space structure of S^1 i.e. the pinching map:

$$S^1 \longrightarrow S^1 \vee S^1.$$

2.3.10. *Intersection of families of closed strings.* Now consider two families of closed strings (P, f) and (Q, g) , by deforming $f \times g$ one can produce a new family of closed strings $(P * Q, \Upsilon\psi)$ in M . We also notice that the image of $\Upsilon\psi$ lies in $\mathcal{L}M^{(0)} \cup \mathcal{L}M^{(1)}$.

2.3.11. **Remark.** All we have done with free loop spaces can be performed for manifolds of maps from a space which is a co-H-space and a compact orientable manifold to a compact Riemannian manifold.

2.4. **n -sphere spaces.** Let M be a n -connected d -dimensional compact oriented smooth manifold.

2.4.1. **Definition.** We call the n -sphere space of M and we denote it by $\mathcal{S}_n M$ the space of \mathbf{H}^n -maps from S^n to M .

2.4.2. **Remark.** By Sobolev's embedding theorem we know that

$$\mathbf{H}^n(S^n, M) \subset C^0(S^n, M).$$

2.4.3. **Theorem.** $\mathcal{S}_n M$ is an Hilbert manifold.

Proof As for free loop spaces an atlas of $\mathcal{S}_n M$ can be given by considering \mathbf{H}^n vector fields along all maps

$$b : S^n \rightarrow M.$$

Then using a trivialization of b^*TM we deduce that $\mathcal{S}_n M$ is modelled on the separable Hilbert space $\mathcal{S}_n \mathbb{R}^d$. \square

The tangent bundle of $\mathcal{S}_n M$ has the same description as $T\mathcal{L}M$. It can be identified with $\mathcal{S}_n TM$ and we have:

$$T\mathcal{S}_n M_b = \Gamma(b^*TM).$$

Moreover, $\mathcal{S}_n M$ is a Riemannian manifold.

2.4.4. *The n -sphere pull-back.* Let fix a base point 0 in S^n , the evaluation map:

$$ev_0 : \mathcal{S}_n M \rightarrow M$$

is clearly a submersion of Hilbert manifolds, then we can form the pull-back of Hilbert manifolds:

$$\begin{array}{ccc} \mathcal{S}_n M \times \mathcal{S}_n M & \xleftarrow{\tilde{\Delta}} & \mathcal{S}_n M \cap_M \mathcal{S}_n M \\ ev_0 \times ev_0 \downarrow & & \downarrow ev \\ M \times M & \xleftarrow{\Delta} & M, \end{array}$$

and the map $\tilde{\Delta}$ is an orientable morphism.

2.4.5. *Composing n -sphere.* Thanks to the pinching map:

$$S^n \longrightarrow S^n \vee S^n$$

one can define:

$$\Upsilon : \mathcal{S}_n M \cap_M \mathcal{S}_n M \longrightarrow \mathcal{S}_n M.$$

2.4.6. *Intersection of families of n -sphere.* As for families of closed strings, we consider two families of n -sphere in M denoted by (P, f) and (Q, g) , by deforming $f \times g$ and taking the pullback $P * Q$ along $\tilde{\Delta}$ one can produce a new family of n -sphere $(P * Q, \Upsilon\psi)$ in M .

3. GEOMETRIC HOMOLOGY THEORIES

As R. Thom proved it is not possible in general to represent singular homology classes of a topological space X by singular maps i.e continuous maps:

$$f : P \longrightarrow X$$

from an oriented manifold to X . But, M. Jakob in [28], [29] proves that if we add a cohomological information to the map f (a singular cohomological class of P), then Steenrod's realizability problem with this additional cohomological data has an affirmative answer. In these two papers he develops a geometric version of homology. This geometric version seems to be very nice to deal with Gysin morphisms, intersection products and so on.

All the constructions we give below and also their applications to string topology work out for more general homology theories: bordism, topological K -theory for example. We refer the reader to [28], [29] and [30] for the definitions of these geometric theories.

3.1. An alternative description of singular homology.

3.1.1. *Geometric cycles.* Let X be a topological space, a geometric cycle is a triple (P, a, f) where:

$$f : P \longrightarrow X$$

is a continuous map from a compact connected orientable manifold P to X (i.e a singular manifold over X), and $a \in H^*(P, \mathbb{Z})$. If P is of dimension p and $a \in H^m(P, \mathbb{Z})$ then (P, a, f) is a geometric cycle of degree $p - m$. Take the free abelian group generated by all the geometric cycles and impose the following relation:

$$(P, \lambda \cdot a + \mu \cdot b, f) = \lambda \cdot (P, a, f) + \mu \cdot (P, b, f).$$

Thus we get a graded abelian group.

3.1.2. *Relations.* In order to recover singular homology we must impose the two following relations on geometric cycles:

i) **(Bordism relation)** If we have a map $h : W \rightarrow X$ where W is an orientable bordism between (P, f) and (Q, g) i.e.

$$\partial W = P \bigcup Q^-.$$

Let $i_1 : P \hookrightarrow W$ and $i_2 : Q \hookrightarrow W$ be the canonical inclusions, then for any $c \in H^*(W, \mathbb{Z})$ we impose:

$$(P, i_1^*(c), f) = (Q, i_2^*(c), g).$$

ii) **(Vector bundle modification)** Let (P, a, f) be a geometric cycle and consider a smooth orientable vector bundle $E \xrightarrow{\pi} P$, take the unit sphere bundle $S(E \oplus 1)$ of the Whitney sum of E with a copy of the trivial bundle over M . The bundle $S(E \oplus 1)$ admits a section σ , by $\sigma_!$ we denote the Gysin morphism in cohomology associated to this section. Then we impose:

$$(P, a, f) = (S(E \oplus 1), \sigma_!(a), f\pi).$$

An equivalence class of geometric cycle is denoted by $[P, a, f]$, let call it a geometric class. And $H'_q(X)$ is the abelian group of geometric classes of degree q .

3.1.3. **Theorem.** [28, Cor. 2.36] *The morphism:*

$$H'_q(X) \longrightarrow H_q(X, \mathbb{Z})$$

$$[P, a, f] \mapsto f_*(a \cap [P])$$

where $[P]$ is the fundamental class of P is an isomorphism of abelian groups.

3.2. Cap product and Poincaré duality [29, 3.2]. The cap product between $H^*(X, \mathbb{Z})$ and $H'_*(X)$ is given by the following formula:

$$\cap : H^p(X, \mathbb{Z}) \otimes H'_q(X) \longrightarrow H'_{q-p}(X)$$

$$u \cap [P, a, f] = [P, f^*(u) \cup a, f].$$

Let M be a d -dimensional smooth compact orientable manifold without boundary then the morphism:

$$H^p(M, \mathbb{Z}) \longrightarrow H'_{d-p}(M)$$

$$x \mapsto [M, x, Id_M]$$

is an isomorphism.

3.3. Gysin morphisms. ([30] for a finite dimensional version) We want to consider Gysin morphisms in the context of infinite dimensional manifolds. Let us recall two possible definitions for Gysin morphisms in the finite dimensional context. The following one is only relevant to the final dimensional case. Let us take a morphism:

$$f : M^m \longrightarrow N^n$$

of Poincaré duality spaces. Then we define:

$$f_! : H_*(N^n) \xrightarrow{D} H^{n-*}(N^n) \xrightarrow{f} H^{n-*}(M^m) \xrightarrow{D^{-1}} H_{*+m-n}(M^m),$$

where D is the Poncaré duality isomorphism.

For the second construction, if f is an embedding of smooth oriented manifold then one can apply the Pontryagin-Thom collapse c to the Thom space of the normal bundle of f and then apply the Thom isomorphism th :

$$f_! : H_*(N^n) \xrightarrow{c} H_*(Th(\nu(f))) \xrightarrow{th} H_{*+m-n}(M^m)$$

In the infinite dimensional context, we can not use Poincaré duality, and to be as explicit as possible we do not want to use the Pontryagin-Thom collapse and the Thom isomorphism. We prefer to use a very geometrical interpretation of the Gysin morphism which is to take pull backs of cycles along the map f .

So, we take $i : X \rightarrow Y$ an orientable morphism of Hilbert manifolds and we suppose that $\nu(i)$ is d -dimensional. Let us define:

$$i^! : H'_p(Y) \longrightarrow H'_{p-d}(X).$$

Let $[P, a, f]$ be a geometric class in $H'_p(Y)$, we can choose a representing cycle (P, a, f) . If f is not smooth, we know that it is homotopic to a smooth map by the existence of partitions of unity on Y , moreover we can choose it

transverse to i , by the bordism relation all these cycles represent the same class. Now we can form the pull-back:

$$\begin{array}{ccc} P & \xleftarrow{f^*i} & P \cap_Y X \\ f \downarrow & & \downarrow \phi \\ Y & \xleftarrow{i} & X \end{array}$$

we set:

$$i^!([P, a, f]) = (-1)^{d \cdot |a|} [P \cap_Y X, (f^*i)^*(a), \phi].$$

The sign is taken from [30, 3.2c)], the Gysin morphism can be viewed as a product for bivariant theories [17].

3.4. The cross product [29, 3.1]. The cross product is given by the pairing:

$$\begin{aligned} \times : H'_q(X) \otimes H'_p(Y) &\longrightarrow H'_{p+q}(X \times Y) \\ [P, a, f] \times [Q, b, g] &= (-1)^{\dim(P) \cdot |b|} [P \times Q, a \times b, f \times g]. \end{aligned}$$

The sign makes the cross product commutative. let

$$\tau : X \times Y \rightarrow Y \times X$$

be the interchanging morphism then:

$$\tau_*(\alpha \times \beta) = (-1)^{|\alpha||\beta|} \beta \times \alpha.$$

3.5. The intersection product ([30, sect.3]). Let us return to the finite dimensional case and consider M an orientable compact d -dimensional manifold. Like for Gysin morphisms in order to be very explicit we avoid the classical constructions of the intersection product that use either Poincaré duality or the Thom isomorphism.

Let $[P, x, f] \in H'_{n_1}(M)$ and $[Q, y, g] \in H'_{n_2}(M)$, we suppose that f and g are transverse in M , then we form the pull back:

$$\begin{array}{ccc} P \times Q & \xleftarrow{j} & P \cap_M Q \\ f \times g \downarrow & & \downarrow \phi \\ M \times M & \xleftarrow{\Delta} & M \end{array}$$

and define the pairing:

$$- \bullet - : H'_{n_1}(M) \otimes H'_{n_2}(M) \xrightarrow{\times} H'_{n_1+n_2}(M \times M) \xrightarrow{\Delta^!} H'_{n_1+n_2-d}(M).$$

Hence, we set:

$$[P, a, f] \bullet [Q, b, g] = (-1)^{d \cdot (|a|+|b|) + \dim(P) \cdot |b|} [P \cap_M Q, j^*(a \times b), \phi].$$

Let $l : P \cap_M Q \rightarrow P$ and $r : P \cap_M Q \rightarrow Q$ be the canonical maps, then we also have:

$$[P, a, f] \bullet [Q, b, g] = (-1)^{d.(|a|+|b|)+\dim(P).|b|} [P \cap_M Q, l^*(a) \cup r^*(b), \phi].$$

With these signs conventions the intersection product \bullet makes $H'_{*+d}(M)$ into a graded commutative algebra:

$$[P \cap_M Q, l^*(a) \cup r^*(b), \phi] = (-1)^{(d-\dim(P)-|a|)(d-\dim(Q)-|b|)} [Q \cap_M P, l^*(b) \cup r^*(a), \phi].$$

4. STRING TOPOLOGY

In this section, using the theory of geometric cycles we show how to recover the BV -structure on

$$\mathbb{H}_*(\mathcal{LM}) := H'_{*+d}(\mathcal{LM}, \mathbb{Z})$$

introduced in [8] and studied from a homotopical point of view in [13]. We also define the intersection morphism, the string bracket of [8] and string topology operations (we extend the Frobenius structure given in [12] to a homological action of the space of Sullivan's chord diagrams).

Remark: In this section we use the language of operads and algebras over an operad (in order to state some results in a nice and appropriate framework). For definitions and examples of operads and algebras over an operad we refer to [21], [23], [38], [40], [41] and [53].

4.1. The operator Δ . First we define the Δ -operator on $\mathbb{H}_*(\mathcal{LM})$. Let us consider a geometric cycle $[P, a, f] \in H'_{n+d}(\mathcal{LM})$, we have a map:

$$\Theta_f : S^1 \times P \xrightarrow{Id \times f} S^1 \times \mathcal{LM} \xrightarrow{\Theta} \mathcal{LM}.$$

4.1.1. Definition. *There is an operator*

$$\Delta : H'_{n+d}(\mathcal{LM}) \rightarrow H'_{n+d+1}(\mathcal{LM})$$

given by the following formula:

$$\Delta([P, a, f]) = (-1)^{|a|} [S^1 \times P, 1 \times a, \Theta_f].$$

4.1.2. Proposition [8, prop. 5.1]. *The operator verifies: $\Delta^2 = 0$.*

Proof This follows from the associativity of the cross product and the nullity of $[S^1 \times S^1, 1 \times 1, \mu] \in H'_2(S^1)$ where μ is the product on S^1 . \square

4.2. Loop product. Let us take $[P, a, f] \in H'_{n_1+d}(\mathcal{LM})$ and $[Q, b, g] \in H'_{n_2+d}(\mathcal{LM})$. We can smooth f and g and make them transverse to $\tilde{\Delta}$, then we form the pull-back $P * Q$.

4.2.1. **Definition.** Let $l : P * Q \rightarrow P$ and $r : P * Q \rightarrow Q$ be the canonical maps, then we have the pairing:

$$-\bullet- : H'_{n_1+d}(\mathcal{L}M) \otimes H'_{n_2+d}(\mathcal{L}M) \longrightarrow H'_{n_1+n_2+d}(\mathcal{L}M)$$

$$[P, a, f] \bullet [Q, b, g] = (-1)^{d.(|a|+|b|)+\dim(P).|b|} [P * Q, l^*(a) \cup r^*(b), \Upsilon\psi],$$

let call it the loop product.

4.2.2. **Proposition** [8, Thm. 3.3]. The loop product is associative and commutative.

Proof The associativity of the loop product follows from the associativity of the intersection product, the cup product and the fact that Υ is also associative up to homotopy.

In order to prove the commutativity of \bullet we follow the strategy of [8, Lemma 3.2].

There is a smooth interchanging map:

$$\tau : \mathcal{L}M \cap_M \mathcal{L}M \rightarrow \mathcal{L}M \cap_M \mathcal{L}M.$$

Let $[P, a, f]$ and $[Q, b, g]$ be two geometric classes the formula of [8, lemma 3.2] gives an homotopy H (so this is also a bordism) between:

$$P * Q \xrightarrow{\psi} \mathcal{L}M \cap_M \mathcal{L}M \xrightarrow{\Upsilon\tau} \mathcal{L}M$$

and

$$P * Q \xrightarrow{\psi} \mathcal{L}M \cap_M \mathcal{L}M \xrightarrow{\Upsilon\tau} \mathcal{L}M.$$

This bordism identifies $[P * Q, l^*(a) \cup r^*(b), \Upsilon\psi]$ and

$$[P * Q, \tau^*(l^*(a) \cup r^*(b)), \Upsilon\tau\psi]$$

which is equal to:

$$(-1)^{(\dim(P)-d-a)(\dim(P)-d-b)} [Q * P, l^*(b) \cup r^*(a), \Upsilon\psi].$$

□

4.3. **Loop bracket.** Let $[P, a, f]$ and $[Q, b, g]$ be two geometric classes, in the preceding section we have defined a bordism between

$$P * Q \xrightarrow{\psi} \mathcal{L}M \cap_M \mathcal{L}M \xrightarrow{\Upsilon\tau} \mathcal{L}M$$

and

$$P * Q \xrightarrow{\psi} \mathcal{L}M \cap_M \mathcal{L}M \xrightarrow{\Upsilon\tau} \mathcal{L}M.$$

Using the same homotopy one can define another bordism between

$$P * Q \xrightarrow{\psi} \mathcal{L}M \cap_M \mathcal{L}M \xrightarrow{\Upsilon\tau^2} \mathcal{L}M$$

and

$$P * Q \xrightarrow{\psi} \mathcal{L}M \cap_M \mathcal{L}M \xrightarrow{\Upsilon\tau^2} \mathcal{L}M.$$

Composing these two bordisms one obtains a geometric class:

$$(-1)^{|a|+|b|} [S^1 \times P * Q, 1 \times l^*(a) \cup r^*(b) \times 1, \tilde{H}].$$

4.3.1. **Definition.** *The loop bracket is the pairing:*

$$\{-, -\} : H'_{n_1+d}(\mathcal{L}M) \otimes H'_{n_2+d}(\mathcal{L}M) \longrightarrow H'_{n_1+n_2+d+1}(\mathcal{L}M)$$

$$\{[P, a, f], [Q, b, g]\} = (-1)^{(d+1)\cdot(|a|+|b|)+\dim(P)\cdot|b|} [S^1 \times P*Q, 1 \times l^*(a) \cup r^*(b), \tilde{H}].$$

There is another way to define the bracket by setting [8, Cor. 5.3]:

$$\{\alpha, \beta\} = (-1)^{|\alpha|} \Delta(\alpha \bullet \beta) - (-1)^{|\alpha|} \Delta(\alpha) \bullet \beta - \alpha \bullet \Delta(\beta).$$

Together with this bracket, $(\mathbb{H}_*(\mathcal{L}M), \bullet, \{-, -\})$ is a Gerstenhaber algebra.

4.3.2. **Theorem.** [8, Thm. 4.7] *The triple $(\mathbb{H}_*(\mathcal{L}M), \bullet, \{-, -\})$ is a Gerstenhaber algebra:*

- i) $(\mathbb{H}_*(\mathcal{L}M), \bullet)$ is a graded associative and commutative algebra.
- ii) The loop bracket $\{-, -\}$ is a Lie bracket of degree +1:

$$\{\alpha, \beta\} = (-1)^{(|\alpha|+1)(|\beta|+1)} \{\beta, \alpha\},$$

$$\{\alpha, \{\beta, \gamma\}\} = \{\{\alpha, \beta\} \cdot \gamma\} + (-1)^{(|\alpha|+1)(|\beta|+1)} \{\beta, \{\alpha, \gamma\}\},$$

$$iii) \{\alpha, \beta \bullet \gamma\} = \{\alpha, \beta\} \bullet \gamma + (-1)^{|\beta|(|\alpha|+1)} \beta \bullet \{\alpha, \gamma\}.$$

4.3.3. **Remark.** Let us recall that there are two important examples of Gerstenhaber algebras:

- The first one is the Hochschild cohomology of a differential graded associative algebra A :

$$HH^*(A, A),$$

this goes back to M. Gerstenhaber [18].

- The second example is the singular homology of a double loop space:

$$H_*(\Omega^2 X),$$

this is due to F. Cohen [11].

Both examples are related by the Deligne's conjecture proved in many different ways by C. Berger and B. Fresse [7], M. Kontsevich and Y. Soibelman [36], J. McClure and J. Smith [42], D. Tamarkin [49] and S. Voronov [52] (see also M. Kontsevich [35]). This conjecture states that there is a natural action of an operad C_2 quasi-isomorphic to the chain operad of little 2-discs on the Hochschild cochain complex of an associative algebra.

Hochschild homology enters the theory by the following results of R. Cohen and J.D.S. Jones [13, Thm. 13]:

if $C^*(M)$ denotes the singular cochains of a manifold M , then there is an isomorphism of associative algebras:

$$HH^*(C^*(M), C^*(M)) \cong \mathbb{H}_*(\mathcal{L}M).$$

4.4. **The BV-structure.** In [13] and [8] it is proved that $\mathbb{H}_*(\mathcal{L}M)$ is a BV-algebra (we refer to [19] for BV-structures).

4.4.1. **Theorem** [8, Th. 5.4]. *The loop product \bullet and the operator Δ makes $\mathbb{H}_*(\mathcal{L}M)$ into a Batalin Vilkovisky algebra, we have the following relations:*

- i) $(\mathbb{H}_*(M^{S^1}), \bullet)$, is a graded commutative associative algebra.
- ii) $\Delta^2 = 0$
- iii) $(-1)^{|\alpha|} \Delta(\alpha \bullet \beta) - (-1)^{|\alpha|} \Delta(\alpha) \bullet \beta - \alpha \bullet \Delta(\beta)$ is a derivation of each variable.

The proof of the theorem in the context of geometric homology is given by building explicit bordisms between geometric cycles. All these bordisms are described in [8].

4.4.2. **Remark.** E. Getzler introduced *BV*-algebras in the context of 2-dimensional topological field theories [19]. And he proved that $H_*(\Omega^2 M)$ is a *BV*-algebra if M has a S^1 action. Other examples are provided by the de Rham cohomology of manifolds with S^1 -action.

The *BV*-structure on $\mathbb{H}_*(\mathcal{L}M)$ comes in fact from a geometric action of the *cacti* operad [13], [53] (normalized cacti with spines in the terminology of R. Kaufmann [34]). Roughly speaking an element of $cacti(n)$ is a tree-like configuration of n -marked circles in the plane. The cacti operad is homotopy equivalent to the little framed discs operad [53]. And we know since the work of E. Getzler that the homology of the little framed discs operad gives the *BV* operad [19].

Let us explain this geometric action. First let us define the space $\mathcal{L}^{cacti(n)} M$ (denoted by $L_k M$ in [13]) as:

$$\mathcal{L}^{cacti(n)} M = \{(c, f) : c \in cacti(n), f : c \rightarrow M\}$$

we take the Gysin morphism along a map:

$$cacti(n) \times \mathcal{L}M^{\times n} \longrightarrow \mathcal{L}^{cacti(n)} M$$

to any element $c \in cacti(n)$ one can associate a map:

$$S^1 \rightarrow c$$

then we get:

$$\mathcal{L}^{cacti(n)} M \longrightarrow \mathcal{L}M.$$

For $n = 1, 2$ we know from R. Kaufmann's description of *cacti* [32] that $cacti(n)$ is a smooth manifold. In that case all the maps defined above are maps of Hilbert manifolds and they give also a very nice description of the action of $H'_*(cacti)$ on $H'_{*+d}(\mathcal{L}M)$.

So, it is certainly worth building a smooth structure on *cacti* or on an operad homotopy equivalent that acts in the same way. This would give a more conceptual proof of the preceding theorem.

4.5. Constant strings. We have a canonical embedding:

$$c : M \hookrightarrow \mathcal{L}M$$

c induces a map:

$$c_* : H'_{n+d}(M) \rightarrow H'_{n+d}(\mathcal{L}M).$$

The morphism c_* is clearly a morphism of commutative algebras.

4.6. Intersection morphism. Let recall that the map

$$ev_0 : \mathcal{L}M \longrightarrow M$$

is a submersion (in fact this is a smooth fiber bundle of Hilbert manifolds). Hence if we choose a base point $m \in M$ the fiber of ev_0 in m is the Hilbert manifold ΩM of based loops in M . Consider the morphism:

$$i : \Omega M \hookrightarrow \mathcal{L}M$$

from the based loops in M to the free loops in M , this is an orientable morphism of codimension d .

Let us describe the intersection morphism:

$$I = i^! : \mathbb{H}_*(\mathcal{L}M) \rightarrow H_*(\Omega M).$$

Let $[P, a, f] \in H'_{n+d}(\mathcal{L}M)$ be a geometric class, one can define $I([P, a, f])$ in two ways:

i) using the Gysin morphism : $I([P, a, f]) = (-1)^{d \cdot |a|} [P \cap_{\mathcal{L}M} \Omega M, (f^* i)^*(a), \phi]$.

A better way is certainly to notice that this is the same as doing the loop product with $[c_m, 1, c]$ where c_m is a point and $c : c_m \rightarrow \mathcal{L}M$ is the constant loop space at the point m , then we have:

ii) $I([P, a, f]) = (-1)^{d \cdot |a|} [P * c_m, l^*(a), \psi]$.

We remark that $P * c_m$ is either empty (depending on the dimension of P , for example when $\dim P < d$) or equal to m . And if $|a| > 0$ we also have $I([P, a, f]) = 0$.

4.6.1. Proposition [8, Prop 3.4]. *The intersection morphism I is a morphism of associative algebras.*

Proof. The algebra structure on $H'_*(\Omega M)$ comes from the Pontryagin product which is the restriction of Υ to $\Omega M \times \Omega M$, we have the following diagram:

$$\begin{array}{ccc} \Omega M \times \Omega M & \xrightarrow{\Upsilon_{\Omega M \times \Omega M}} & \Omega M \\ i \times i \downarrow & & \downarrow i \\ \mathcal{L}M \cap_M \mathcal{L}M & \xrightarrow{\Upsilon} & \mathcal{L}M. \end{array}$$

The Pontryagin product is given by the formula:

$$[P, a, f].[Q, b, g] = (-1)^{\dim(P) \cdot |b|} [P \times Q, a \times b, \Upsilon_{\Omega M \times \Omega M}(f \times g)].$$

This product is associative but not commutative. The intersection morphism is a morphism of algebras by commutativity of the diagram above. \square

This morphism has been studied in details in [16], in particular it is proved that the kernel of I is nilpotent.

4.7. Bordism and string topology. Let $\Omega_*^{SO}(X)$ be the bordism group of a topological space X , we recall that it is isomorphic to the bordism classes of singular oriented manifolds over X (morphism $f : M \rightarrow X$).

We remark that $\Omega_*^{SO}(\mathcal{LM})$ is also a BV -algebra (all the constructions described above immediately adapt to Ω_*^{SO}).

Let call a geometric class $[P, a, f]$ realizable if it is equivalent to a class $[Q, 1, g]$. This is equivalent to condition of being in the image of the Steenrod-Thom morphism:

$$\begin{aligned} st : \Omega_{n+d}^{SO}(\mathcal{LM}) &\rightarrow \mathbb{H}_n(M) \\ [M, f] &\mapsto [M, 1, f]. \end{aligned}$$

This morphism is clearly a morphism of BV -algebras and we have:

4.7.1. Proposition. *If $c \notin \text{Im}(st)$ then $I(c) = 0$.*

Proof This follows from the fact that if a geometric class is not realizable (via the morphism st) it has the form $[P, a, f]$ with $|a| > 0$ and in this case $I([P, a, f]) = 0$. \square

4.7.2. Remark: We recall that in general st is neither injective nor surjective. However it is surjective when $n + d < 6$ (using Atiyah-Hirzebruch spectral sequence one proves that it is an isomorphism for $n + d = 0, 1, 2$) and it is also surjective if we work over \mathbb{F}_2 , over this field the orientability condition in the definition of geometric homology is unnecessary.

4.8. String bracket.

4.8.1. The string space. Let us consider the fibration:

$$S^1 \rightarrow ES^1 \rightarrow BS^1.$$

There exists a smooth model for this fibration, for ES^1 we take S^∞ the inductive limit of S^n . By [37, chapter X] this is an Hilbert manifold modelled on $\mathbb{R}^{(\mathbb{N})}$. As S^1 acts freely and smoothly on S^∞ we have a S^1 fiber bundle of Hilbert manifolds:

$$S^1 \rightarrow S^\infty \xrightarrow{\pi} \mathbb{C}P^\infty.$$

We get the S^1 -fibration:

$$S^1 \rightarrow \mathcal{LM} \times S^\infty \rightarrow \mathcal{LM} \times_{S^1} S^\infty.$$

The projection:

$$\mathcal{LM} \times S^\infty \rightarrow \mathcal{LM}$$

is a homotopy equivalence of Hilbert manifolds. As we know from [15] that an homotopy equivalence between two separable Hilbert manifolds is homotopic to a diffeomorphism, we deduce that they are diffeomorphic.

The space $\mathcal{LM} \times_{S^1} S^\infty$ is not an Hilbert manifold because the action of S^1 on \mathcal{LM} is not smooth. Let call this space the string space of M .

4.8.2. String homology. Let \mathcal{H}_i be the homology group $H'_{i+d}(\mathcal{LM} \times_{S^1} S^\infty)$, this is the string homology of M . In what follows we give explicit definitions of the morphism c , M , E of [8, 6].

The morphism c . Let $e \in H^2(\mathcal{LM} \times_{S^1} S^\infty)$ be the Euler class of the S^1 -fibration defined above:

$$\begin{aligned} c : \mathcal{H}_i &\rightarrow \mathcal{H}_{i-2} \\ c([P, a, f]) &= [P, f^*(e) \cup a, f]. \end{aligned}$$

The morphism E . This morphism is $E = \pi_*$:

$$\begin{aligned} E : \mathbb{H}_i(\mathcal{LM}) &\rightarrow \mathcal{H}_i \\ E([P, a, f]) &= [P, a, \pi f]. \end{aligned}$$

The morphism M . Let $[P, a, f]$ a geometric class in \mathcal{H}_i . For a point $p \in P$, we can choose $(c_p, u) \in \mathcal{LM} \times S^\infty$ that represents a class $[c_p, u] \in \mathcal{LM} \times_{S^1} S^\infty$, as this choice is non-canonical we take all the orbit of (c_p, u) under the action of S^1 . In this way we produce a map

$$\tilde{f} : S^1 \times P \rightarrow \mathcal{LM} \times S^\infty.$$

Identifying $\mathcal{LM} \times S^\infty$ with \mathcal{LM} one get a map:

$$\begin{aligned} M : \mathcal{H}_i &\rightarrow \mathbb{H}_{i+1}(\mathcal{LM}) \\ M([P, a, f]) &= (-1)^{|a|} [S^1 \times P, 1 \times a, \tilde{f}]. \end{aligned}$$

We have the following exact sequence, which is the Gysin exact sequence associated to the S^1 -fibration π :

$$\dots \rightarrow \mathbb{H}_i(\mathcal{LM}) \xrightarrow{E} \mathcal{H}_i \xrightarrow{c} \mathcal{H}_{i-2} \xrightarrow{M} \mathbb{H}_{i-1}(\mathcal{LM}) \rightarrow \dots$$

4.8.3. The bracket. The string bracket is given by the formula:

$$[\alpha, \beta] = (-1)^{|a|} E(M(\alpha) \bullet M(\beta)).$$

Together with this bracket $(\mathcal{H}_*, [-, -])$ is a graded Lie algebra of degree $(2-d)$ [8, Th. 6.1].

4.9. Riemann surfaces operations. These operations are defined by R. Cohen and V. Godin in [12] by means of Thom spectra technology.

Let Σ be an oriented surface of genus g with $p + q$ boundary components, p incoming and q outgoing. We fix a parametrization of these components. Hence we have two maps:

$$\rho_{in} : \coprod_p S^1 \rightarrow \Sigma,$$

and

$$\rho_{out} : \coprod_q S^1 \rightarrow \Sigma.$$

If we consider the space of \mathbf{H}^2 -maps $\mathbf{H}^2(\Sigma, M)$, we get a Hilbert manifold and the diagram of Hilbert manifolds:

$$\mathcal{L}M^{\times q} \xleftarrow{\rho_{out}} \mathbf{H}^2(\Sigma, M) \xrightarrow{\rho_{in}} \mathcal{L}M^{\times p}.$$

Let $\chi(\Sigma)$ be the Euler characteristic of the surface. Using Sullivan's Chord diagrams it is proved in [12] that the morphism

$$\mathbf{H}^2(\Sigma, M) \xrightarrow{\rho_{in}} \mathcal{L}M^{\times p}$$

has a homotopy model:

$$\mathbf{H}^2(c, M) \xrightarrow{\rho_{in}} \mathcal{L}M^{\times p}$$

that is an embedding of Hilbert manifolds of codimension $-\chi(\Sigma).d$. Hence by using the Gysin morphism for Hilbert manifolds one can define the operation:

$$\mu_\Sigma : H'_*(\mathcal{L}M^{\times p}) \xrightarrow{\rho_{in}^!} H'_{*+\chi(\Sigma).d}(\mathbf{H}^2(\Sigma, M)) \xrightarrow{\rho_{out*}} H'_{*+\chi(\Sigma).d}(\mathcal{L}M^{\times q}).$$

All these operations are parametrized by the topological space of marked, metric chord diagrams $\mathcal{CF}_{p,q}^\mu(g)$ [12, sect1]. In the next section we introduce a reduced version of Sullivan's chord diagrams and give some algebraic properties of the associated operations.

4.9.1. Sullivan's chord diagrams. In the preceding morphism $c \in \mathcal{CF}_{p,q}^\mu(g)$ is the *Sullivan's chord diagram* associated to the surface Σ . Let us recall the definition of [12]:

4.9.2. Definition. *A metric marked Sullivan chord diagram c of type $(g;p,q)$ is a metric fat graph:*

- a graph whose vertices are at least trivalent such that the incoming edges are equipped with a cyclic ordering,

- it has the structure of a compact metric space (details are given in [12, def. 1] and [27, chapter 8]),

this fat graph represents a surface of genus g with $p + q$ boundary components. The set of metric fat graphs is denoted by $\mathcal{F}_{p,q}(g)$.

The cycling ordering of the edges defines "boundary cycles". Pick an edge and an orientation on it, then traverse it in the direction of the orientation,

this leads to a vertex, at this vertex take the next edge coming from the cycling ordering and so on. Then we get a cycle in the set of edges.

The graph c consists of a union of p parametrized circles of varying radii that represent the incoming boundary components, joined at a finite number of points.

Each boundary cycle has a marking. This marking correspond to the starting point of a S^1 -parametrization.

Fat graphs (also called ribbon graphs) are a nice combinatoric tool in order to study Riemann surfaces [44], [45] and [48].

Following a suggestion of A. Voronov rather than using the space $C\mathcal{F}_{p,q}^\mu(g)$, we introduce the space of reduced metric marked Sullivan chord diagrams denoted by $\overline{C\mathcal{F}_{p,q}^\mu(g)}$. In a Sullivan diagram there are ghost edges, these are the edges that lie on the p disjoint circles. Hence there is a continuous map:

$$\pi : c \mapsto S(c)$$

that collapses each ghost edge to a vertex, let us notice that $S(c)$ is also a metric fat graph but it is no more a Sullivan's Chord diagram. And let $\overline{C\mathcal{F}_{p,q}^\mu(g)}$ be the image of π . The space $\overline{C\mathcal{F}_{p,q}^\mu(g)}$ has the following properties:

4.9.3. Proposition. *Let $i : C\mathcal{F}_{p,q}^\mu(g) \rightarrow \mathcal{F}\text{at}_{p,q}(g)$ be the canonical inclusion then i and π are homotopic.*

4.9.4. Proposition. *The space $\overline{C\mathcal{F}_{p,q}^\mu(g)}$ is a Prop.*

Proof. The structure of Prop is obtained via a gluing procedure by identifying an incoming boundary component with an outgoing boundary component. This procedure is described at the level of $C\mathcal{F}_{p,q}^\mu(g)$ in [12]. On unreduced diagrams this procedure is not well defined because there is an ambiguity coming from ghost edges, this is no more the case at the level of $\overline{C\mathcal{F}_{p,q}^\mu(g)}$. \square

4.9.5. Proposition. *The suboperads $\overline{C\mathcal{F}_{*,1}^\mu(0)}$ and $\overline{C\mathcal{F}_{1,*}^\mu(0)}$ are isomorphic to the cacti operad.*

Proof. This is immediate. Let us notice that any element in the preimage $S^{-1}(\overline{c})$ where

$$S : C\mathcal{F}_{1,q}^\mu(0) \rightarrow \overline{C\mathcal{F}_{1,q}^\mu(0)}$$

is a chord diagram associated to the cactus \overline{c} as defined in [34]. \square

4.9.6. Theorem. *For $q > 0$ we have morphisms:*

$$\mu_{n,p,q}(g) : H'_n(\overline{C\mathcal{F}_{p,q}^\mu(g)}) \rightarrow \text{Hom}(H'_*(\mathcal{L}M^{\times p}), H'_{*+\chi(\Sigma).d+n}(\mathcal{L}M^{\times q}))$$

Proof Let us give the construction of $\mu_{n,p,q}(g)$:

consider an element of $H'_n(\overline{\mathcal{CF}_{p,q}^\mu(g)})$ and suppose that is represented by a geometric cycle (S, α, g) where $g : S \rightarrow \overline{\mathcal{CF}_{p,q}^\mu(g)}$. And let us define:

$$Map(g, M) = \{(s, f) / s \in S, f \in Map(g(s), M)\},$$

we also have maps:

$$\rho_{in} : Map(g, M) \rightarrow \mathcal{LM}^{\times p},$$

$$\rho_{out} : Map(g, M) \rightarrow \mathcal{LM}^{\times q}.$$

Using [12, lemma3] we get an embedding of codimension $-\chi(\Sigma).d$:

$$\rho_{in} \times p : Map(g, M) \rightarrow \mathcal{LM}^{\times p} \times S$$

where p is the canonical projection. If this map is an embedding of Hilbert manifolds (we actually don't know if $\overline{\mathcal{CF}_{p,q}^\mu(g)}$ is a manifold), we can form the following diagram:

$$\begin{array}{ccc} N_1 \times \dots \times N_p \times S & \xleftarrow{j} & N_1 * \dots * N_p * S \\ f_1 \times \dots \times f_p \times Id_S \downarrow & & \downarrow \phi_g \\ \mathcal{LM}^{\times p} \times S & \xleftarrow{\rho_{in} \times p} & Map(g, M) \xrightarrow{\rho_{out}} \mathcal{LM}^{\times q}. \end{array}$$

We define $\mu_{n,p,q}$ in the following way:

$$\begin{array}{c} [N_1, \alpha_1, f_1] \otimes \dots \otimes [N_p, \alpha_p, f_p] \\ \downarrow \\ \pm [N_1 \times \dots \times N_p \times S, \alpha_1 \times \dots \times \alpha_p \times \alpha, f_1 \times \dots \times f_p \times id_S] \\ \downarrow \\ \pm [N_1 * \dots * N_p * S, j^*(\alpha_1 \times \dots \times \alpha_p \times \alpha), \rho_{out} \phi_g]. \end{array}$$

Let us define the morphism in a less heuristic and more rigorous way. Rather than using an hypothetical embedding of Hilbert manifolds let use the Thom collapse map of [12, lemma5]:

$$\tau_g : \mathcal{LM}^{\times p} \times S \rightarrow (Map(g, M))^{\nu(g)}$$

where $\nu(g)$ is an open neighborhood of the embedding $\rho_{in} \times p$ and let th_g be the Thom isomorphism:

$$th_g : H'_*((Map(g, M))^{\nu(g)}) \rightarrow H'_{*+\chi(\Sigma).d}(Map(g, M)).$$

Then $\mu_{n,p,q}$ is defined by:

$$\begin{array}{c}
 [N_1, \alpha_1, f_1] \otimes \dots \otimes [N_p, \alpha_p, f_p] \\
 \downarrow \\
 \pm [N_1 \times \dots \times N_p \times S, \alpha_1 \times \dots \times \alpha_p \times \alpha, f_1 \times \dots \times f_p \times id_S] \\
 \downarrow \\
 \pm [N_1 \times \dots \times N_p \times S, \alpha_1 \times \dots \times \alpha_p \times \alpha, \tau_g(f_1 \times \dots \times f_p \times id_S)] \\
 \downarrow \\
 \pm \rho_{out*} th_g([N_1 \times \dots \times N_p \times S, \alpha_1 \times \dots \times \alpha_p \times \alpha, \tau_g(f_1 \times \dots \times f_p \times id_S)]).
 \end{array}$$

□

4.9.7. **Proposition.** $H'_*(\mathcal{LM})$ is an algebra over the prop $H'_*(\overline{CF_{p,q}^\mu(g)})$.

Proof This follows immediately from [12, Cor. 9]. □

This last result unifies some of the algebraic structure arising in string topology, as corollaries we get:

4.9.8. **Corollary.** If we fix $n = 0$, then the action of homological degree 0 string topology operations induces on $H'_{*+d}(\mathcal{LM})$ a structure of Frobenius algebra without co-unit.

4.9.9. **Corollary.** When restricted to $H'_*(\overline{CF_{p,1}^\mu(0)})$ on recovers the BV-structure on $H'_{*+d}(\mathcal{LM})$ induced by the string product and the operator Δ .

4.9.10. **A conjecture and its consequences.** We obtain homological string topology operations which are 4-graded, 3 gradings being purely geometric (p, q and g) and one grading purely homological.

We do not know the homotopy type of $\overline{CF_{p,1}^\mu(0)}$. In fact there is a conjecture of R. Cohen and V. Godin on the topology of the space $CF_{p,q}^\mu$, this conjecture states that this space is homotopy equivalent to $B\mathcal{M}_g^{p+q}$, where \mathcal{M}_g^{p+q} is the mapping class group of a Riemann surface S of genus g with $p+q$ boundary components:

$$\mathcal{M}_q^{p+q} = \pi_0(Diff^+(S, \delta))$$

where the diffeomorphisms must preserve the boundary components pointwise.

Now suppose that this conjecture holds, and let fix $q = 1$, since Tillmann's

work [50] we know that the space $B\mathcal{M}_g^{p+1}$ is an operad $B\mathcal{M}$ and that this operad detects infinite loop spaces. Let Γ be the symmetric operad [3], we have a map of operads:

$$\Gamma \rightarrow B\mathcal{M}.$$

This last fact has a deep consequence in string topology because if we look at the singular homology of the free loop spaces with coefficients in $\mathbb{Z}/p\mathbb{Z}$ this conjecture implies the existence of "stringy Dyer-lashof" operations coming from the symmetries of the surfaces. From the operadic nature of these operations they should satisfy Cartan and Adem relations, and these operations should come from an E_∞ structure [38, chapter 1].

5. n -SPHERE TOPOLOGY.

Let fix an integer $n > 1$, and suppose that M is a n -connected compact oriented smooth manifold. Some results about the algebraic structure of the homology of n -sphere spaces were announced in [53, Th. 2.5]. We show how to recover a part of this structure.

5.1. Sphere product. Let us take $[P, a, f] \in H'_{n_1+d}(\mathcal{S}_n M)$ and $[Q, b, g] \in H'_{n_2+d}(\mathcal{S}_n M)$ two families of n -sphere. We can smoothen f and g and make them transverse to $\tilde{\Delta}$, then we form the pull-back $P * Q$.

5.1.1. Definition. Let $l : P * Q \rightarrow P$ and $r : P * Q \rightarrow Q$ be the canonical maps, then we have the pairing:

$$-\bullet- : H'_{n_1+d}(\mathcal{S}_n M) \otimes H'_{n_2+d}(\mathcal{S}_n M) \longrightarrow H'_{n_1+n_2+d}(\mathcal{S}_n M)$$

$$[P, a, f] \bullet [Q, b, g] = (-1)^{d \cdot (|a|+|b|) + \dim(P) \cdot |b|} [P * Q, l^*(a) \cup r^*(b), \Upsilon \psi],$$

let call it the *sphere product*.

5.1.2. Proposition. The sphere product is associative and commutative.

Proof The associativity and commutativity of the sphere product follows from the associativity and commutativity of the intersection product, the cup product and the fact that Υ is also associative and commutative up to homotopy. \square

5.2. Constant spheres. We have a canonical embedding:

$$c : M \hookrightarrow \mathcal{S}_n M$$

c induces a map:

$$c_* : H'_{*+d}(M) \rightarrow H'_{*+d}(\mathcal{S}_n M).$$

The morphism c_* is clearly a morphism of commutative algebras.

5.3. Intersection morphism. Let us recall that the map

$$ev_0 : \mathcal{S}_n M \longrightarrow M$$

is a submersion (in fact this is a smooth fiber bundle of Hilbert manifolds). Hence if we choose a base point $m \in M$ the fiber of ev_0 in m is the Hilbert manifold $\Omega^n M$ of n -iterated based loops in M . Consider the morphism:

$$i : \Omega^n M \hookrightarrow \mathcal{S}_n M$$

this is an orientable morphism of codimension d .

Let us describe the intersection morphism:

$$I = i^! : H'_{*+d}(\mathcal{S}_n M) \rightarrow H'_*(\Omega M).$$

Let $[P, a, f] \in H'_{*+d}(\mathcal{S}_n M)$ be a geometric class, one can define $I([P, a, f])$ in the following way:

take the sphere product with $[c_m, 1, c]$ where c_m is a point and $c : c_m \rightarrow \mathcal{S}_n M$ is the constant sphere at the point m , then we set:

$$I([P, a, f]) = (-1)^{d \cdot |a|} [P * c_m, l^*(a), \psi].$$

As in the case $n = 1$, we remark we have:

5.3.1. Proposition. *The intersection morphism I is a morphism of graded commutative and associative algebras.*

5.4. 3-sphere topology. Let emphasis on the case $n = 3$, in this case we show the existence of an operator of degree 3 acting on the homology. And we also obtain some results about the S^3 -equivariant homology of 3-sphere spaces.

5.4.1. The operator Δ_3 . The sphere S^3 acts on $\mathcal{S}_3 M$, let denote this action by Θ . Hence, if we consider a family of 3-sphere in M

$$f : P \rightarrow \mathcal{S}_3 M$$

we can build a new family:

$$\Theta_f : S^3 \times P \xrightarrow{id \times f} S^3 \times \mathcal{S}_3 M \xrightarrow{\Theta} \mathcal{S}_3 M.$$

5.4.2. Definition. *The operator Δ_3 is given by the following formula:*

$$\Delta_3 : H'_*(\mathcal{S}_3 M) \rightarrow H'_{*+3}(\mathcal{S}_3 M)$$

$$[P, a, f] \mapsto (-1)^{|a|} [S^3 \times P, 1 \times a, \Theta_f].$$

5.4.3. **Proposition.** *The operator verifies:*

$$\Delta_3^2 = 0.$$

Proof The proof is exactly the same as in the case of the operator Δ and follows from the associativity of the cross product and the nullity of the geometric class:

$$[S^3 \times S^3, 1 \times 1, \mu].$$

□

5.4.4. *3-sphere bracket.* As in the case $n = 1$, we have a smooth model of the S^3 -fibration:

$$S^3 \rightarrow ES^3 \rightarrow BS^3$$

which is given by the S^3 fiber bundle of Hilbert manifolds:

$$S^3 \rightarrow S^\infty \rightarrow \mathbb{H}P^\infty.$$

We consider the S^3 -fibration:

$$S^3 \rightarrow \mathcal{S}_3 M \times ES^3 \rightarrow \mathcal{S}_3 M \times_{S^3} ES^3.$$

Let $\mathcal{H}_*^3 = H_{*+d}(\mathcal{S}_3 M \times ES^3)$ and consider the Gysin exact sequence of the fibration:

$$\dots \rightarrow H'_{i+d}(\mathcal{S}_3 M) \xrightarrow{E} \mathcal{H}_i^3 \xrightarrow{c} \mathcal{H}_{i-4}^3 \xrightarrow{M} H'_{i+d-1}(\mathcal{S}_3 M) \rightarrow \dots$$

5.4.5. **Definition.** *We define the 3-sphere bracket by the formula:*

$$[\alpha, \beta] = (-1)^{|\alpha|} E(M(\alpha) \bullet M(\beta)).$$

5.4.6. **Remark.** This bracket is anti-commutative.

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